# Bifurcation of periodic solutions from a ring configuration of discrete nonlinear oscillators

C. García-Azpeitia
J. Ize
Depto. Matemáticas y Mecánica, IIMAS-UNAM, FENOMEC,
Apdo. Postal 20-726, 01000 México D.F.
cgazpe@hotmail.com

March 28, 2013

**Abstract**

This paper gives an analysis of the periodic solutions of a ring of $n$ oscillators coupled to their neighbors. We prove the bifurcation of branches of such solutions from a relative equilibrium, and we study their symmetries. We give complete results for a cubic Schrödinger potential and for a saturable potential and for intervals of the amplitude of the equilibrium. The tools for the analysis are the orthogonal degree and representation of groups. The bifurcation of relative equilibria was given in a previous paper.

## 1 Introduction

Consider a lattice of $n$ nonlinear oscillators, with a periodicity condition or a ring of $n$ oscillators. The lattice may come from the discretization of a nonlinear PDE, such as a nonlinear Schrödinger equation, with different types of potentials. These equations have been used very often as models of several phenomena of physics of waves, in particular in nonlinear optics, see [2].

In previous papers, we have studied the problems of bifurcation of relative equilibria (see [4]) and bifurcation of periodic solutions for rings of masses ([5]), vortices and filaments ([6]).

Although these problems are different as models of physical phenomena, they have in common several mathematical aspects, such as a similar linearization and the same type of symmetries.

A relative equilibrium is a stationary solution of the system in a constant rotating frame. In particular, we consider one where all oscillators have the same amplitude but different phases. When these equilibria are localized they

are called breathers. In case of a periodic localized solution, one has a quasi-periodic breather, see [2] or [9].

Numerical studies of the bifurcation of relative equilibria for a circular lattice and a defect in one of the oscillators, for $n = 7$, is given in [12]. For the case of a Dirichlet condition, instead of the periodic problem, one has a very complete numerical analysis of bifurcation of breathers in [11].

The linearization of the system at a critical point is a $2n \times 2n$ matrix, which is non invertible, due to the rotational symmetry. These facts imply that the study of the spectrum of the linearization is not an easy task and that the classical bifurcation results for periodic solutions may not be applied directly. However, we shall use the change of variables proved in our previous paper, [4], in order to give not only this spectrum but also the consequences for the symmetries of the solutions.

The present paper is a continuation of [4]. Thus, we shall use the results in that paper, but we shall recall all the important notions. In our parallel papers, [5] and [6], we study similar problems for point masses and vortices. Although there are many similarities, in particular in the change of variables, the results are of a quite different nature.

The next section is devoted to the mathematical setting of the problem, with the symmetries involved. Then, we give, in the following two sections, the preliminary results needed in order to apply the orthogonal degree theory developed in [8], that is the global Liapunov-Schmidt reduction, the study of the irreducible representations, with the change of variables of [4], and the symmetries associated to these representations. In the next section, we prove our bifurcation results and, in the following section, we give the analysis of the spectrum, with the complete results on the type of solutions which bifurcate from the relative equilibrium.

# 2 Setting the problem

Let us denote by $q_j(t) \in \mathbb{C}$ the $j$'th oscillator, for $j \in \{1, ..., n\}$. The dNLS equations are

$$i\dot{q}_j = h(\|q_j\|^2)q_j + (q_{j+1} - 2q_j + q_{j-1}),$$

where $h$ is the nonlinear potential. We wish to study a finite circular lattice, that is, a lattice of oscillators for $j \in \{1, ..., n\}$, with periodic conditions $q_j = q_{j+n}$.

The solutions of the form $q_j = e^{\omega t i}u_j$, with $u_j$ constant, are called relative equilibria. In order to obtain the amplitude as a parameter, we need to change coordinates, with $q_j = \mu e^{\omega t i}u_j$. In this manner, we have that the values $u_j$ form a relative equilibrium when

$$-\omega u_j = h(|\mu u_j|^2)u_j + (u_{j+1} - 2u_j + u_{j-1}).$$

**Remark 1** *Given that the lattice is integrable for $n = 1$ and $n = 2$, we shall look for bifurcation for $n \geq 3$.*

The starting point is a relative equilibrium which looks like a rotating wave. We give a condition which needs to be satisfied by the potential for the existence of this rotating wave.

**Proposition 2** *Define $a_j = e^{ij\zeta}$, with $\zeta = 2\pi/n$, then $\bar{a} = (a_1, ..., a_n)$ is a relative equilibrium if*

$$\omega = 4\sin^2(\zeta/2) - h(\mu^2).$$

The proof is simple and given in [4].

For the study of periodic solutions, we need to change to rotating coordinates: $q_j = \mu e^{\omega t i} u_j$, and the equation becomes

$$i\dot{u}_j - \omega u_j = h(|\mu u_j|^2)u_j + (u_{j+1} - 2u_j + u_{j-1}).$$

Changing to real coordinates, one has, for $u_j \in \mathbb{R}^2$

$$J\dot{u}_j = \omega u_j + h(|\mu u_j|^2)u_j + (u_{j+1} - 2u_j + u_{j-1}),$$

where $J$ is the standard symplectic matrix.

Set $u = (u_1, ..., u_n)^T$ be the vector of positions and let $\mathcal{J} = diag(J, ..., J)$, then the vectorial form of the equation is

$$\mathcal{J}\dot{u} = \nabla V(u) \text{ with}$$
$$V = \frac{1}{2}\sum_{j=1}^{n}\left\{H(u_j) - |u_{j+1} - u_j|^2\right\},$$

where $H(u)$ is the function which satisfies $\nabla H(u) = \omega u + h(|\mu u|^2)u$.

Since we are looking for periodic solutions, let us define $x(t) = u(t/\nu)$. Then, $2\pi/\nu$ solutions for $u$ are $2\pi$ solutions for $x$ of the equation

$$f(x) = -\nu\mathcal{J}\dot{x} + \nabla V(x) = 0.$$

The operator $f$ is defined from $H^1_{2\pi}(\mathbb{R}^{2n})$ into $L^2_{2\pi}(\mathbb{R}^{2n})$.

**Definition 3** *Let $S_n$ be the group of permutations of $\{1, ..., n\}$. One defines the action of $S_n$ in $\mathbb{R}^{2n}$ as*

$$\rho(\gamma)(x_1, ..., x_n) = (x_{\gamma(1)}, ..., x_{\gamma(n)}),$$

*and the action of $\theta \in SO(2)$ as*

$$\rho(\theta)x = e^{-\mathcal{J}\theta}x.$$

Let $\mathbb{Z}_n$ be the subgroup of permutations generated by $\zeta(j) = j + 1$ modulus $n$. The gradient $\nabla V$ is $\mathbb{Z}_n \times SO(2)$-equivariant, that is it commutes with the action of the group, and the map $f$ is $\Gamma \times S^1$-equivariant with the abelian group

$$\Gamma = \mathbb{Z}_n \times SO(2),$$

where the action of $S^1$ is by time translation.

Now, the infinitesimal generators of $S^1$ and $\Gamma$ are

$$Ax = \frac{\partial}{\partial\varphi}|_{\varphi=0}x(t+\varphi) = \dot{x} \text{ and } A_1 x = \frac{\partial}{\partial\theta}|_{\theta=0}e^{-\mathcal{J}\theta}x = -\mathcal{J}x.$$

Since $V$ is $\Gamma$-invariant, then the gradient $\nabla V(x)$ must be orthogonal to the generator $A_1 x$. As a consequence, the map $f$ must be $\Gamma \times S^1$-orthogonal, due to the equalities

$$\begin{aligned}\langle f(x), \dot{x}\rangle &= -\nu \langle \mathcal{J}\dot{x}, \dot{x}\rangle + V(x)|_0^{2\pi} = 0\\ \langle f(x), -\mathcal{J}x\rangle &= \nu\frac{1}{2}\langle x, x\rangle |_0^{2\pi} - \langle \mathcal{J}x, \nabla V(x)\rangle = 0.\end{aligned}$$

Define $\tilde{\mathbb{Z}}_n$ as the subgroup of $\Gamma$ generated by $(\zeta,\zeta) \in \mathbb{Z}_n \times SO(2)$ with $\zeta = 2\pi/n \in SO(2)$. Since the action of $(\zeta,\zeta)$ leaves fixed the equilibrium $\bar{a}$, then the isotropy group of $\bar{a}$ is the group $\Gamma_{\bar{a}} \times S^1$ with

$$\Gamma_{\bar{a}} = \tilde{\mathbb{Z}}_n.$$

Thus, the orbit of $\bar{a}$ is isomorphic to the group $SO(2)$. In fact, the orbit consists of the rotations of the equilibrium. As a consequence, the generator of the orbit $A_1\bar{a} = -\mathcal{J}\bar{a}$ must be in the kernel of $D^2 f(\bar{a})$.

# 3 The Liapunov-Schmidt reduction

In order to apply the orthogonal degree of [8], one needs to make a reduction of the bifurcation map to some finite space.

The bifurcation map $f$ has Fourier series

$$f(x) = \sum_{l\in\mathbb{Z}}(-l\nu i\mathcal{J}x_l + g_l)e^{ilt},$$

where $x_l$ and $g_l$ are the Fourier modes of $x$ and $\nabla V(x)$. Since $-il\nu(i\mathcal{J})$ is invertible for all big $l\nu$'s, then one may solve $x_l$ for $|l| > p$ and $\nu$ bounded from below, whenever $x(t)$ belongs to a bounded set $\Omega$ in the space $H^1$, that is also bounded uniformly in $\mathbb{R}^2$.

In this way, the bifurcation operator $f$ has the same zeros as the bifurcation function

$$f(x_1, x_2(x_1,\nu), \nu) = \sum_{|l|\le p}(-l\nu i\mathcal{J}x_l + g_l)e^{ilt},$$

and the linearization of the bifurcation function at some equilibrium $\bar{a}$ is

$$f'(\bar{a})x_1 = \sum_{|l|\le p}\left(-l\nu i\mathcal{J} + D^2V(\bar{a})\right)x_l e^{ilt}.$$

Here $x_1$ corresponds to the $2p+1$ first Fourier modes and $x_2(x_1, \nu)$ is the result of applying the global implicit function theorem for functions in $\Omega$ and $\nu$ bounded from below.

The linearization of the bifurcation function is determined by blocks $M(l\nu)$ for $l \in \{0, ..., p\}$, where $M(\nu)$ is the matrix

$$M(\nu) = -\nu i\mathcal{J} + D^2V(\bar{a}).$$

These blocks $M(l\nu)$ represent the Fourier modes of the linearized equation at the equilibrium.

# 4 Irreducible representations

In order to apply the orthogonal degree, one needs to find the irreducible representation subspaces for the action of $\Gamma_{\bar{a}} = \tilde{\mathbb{Z}}_n$.

For $k \in \{1, ..., n\}$, we define the isomorphisms $T_k : \mathbb{C}^2 \to W_k$ as

$$T_k(w) = (n^{-1/2}e^{(ikI+J)\zeta}w, ..., n^{-1/2}e^{n(ikI+J)\zeta}w) \text{ with}$$
$$W_k = \{(e^{(ikI+J)\zeta}w, ..., e^{n(ikI+J)\zeta}w) : w \in \mathbb{C}^2\}.$$

In the paper [4], we have proved that the subspaces $W_k$ are irreducible representations of the group $\tilde{\mathbb{Z}}_n$. Also, we showed that the action of $(\zeta, \zeta) \in \tilde{\mathbb{Z}}_n$ on the space $W_k$ is given by

$$\rho(\zeta, \zeta) = e^{ik\zeta}.$$

Since the subspaces $W_k$ are orthogonal, then the linear map

$$Pw = \sum_{j=1}^{n} T(w_k)$$

is orthogonal, where $w = (w_1, ..., w_n)$.

Since the map $P$ rearranges the coordinates of the irreducible representations, one has, from Schur's lemma, that

$$P^{-1}D^2V(\bar{a})P = diag(B_1, ..., B_n),$$

where $B_k$ are matrices which satisfy $D^2V(\bar{a})T_k(w) = T_k(B_k w)$. In the paper [4], we have found the blocks $B_k$: they satisfy $B_{n-k} = \bar{B}_k$ and we had the following result:

Define $\alpha_k$ and $\gamma_k$ as

$$\alpha_k = 4\cos\zeta \sin^2 k\zeta/2 \text{ and } \gamma_k = 2\sin k\zeta \sin\zeta.$$

Then, the blocks $B_k$ are

$$B_k = -\alpha_k I + \gamma_k(iJ) + 2\mu^2 h'(|\mu|^2)diag(1, 0).$$

For the linearization of the equation one has that

$$P^{-1}M(\nu)P = diag(m_1(\nu), ..., m_n(\nu)).$$

Thus, we find the matrices $m_k(\nu)$ in terms of the blocks $B_k$ as

$$m_k(\nu) = -\nu(iJ) + B_k \text{ for } k \in \{1, ..., n\}.$$

The action of $(\zeta, \zeta, \varphi) \in \tilde{\mathbb{Z}}_n \times S^1$ on $W_k$ is $\rho(\zeta, \zeta, \varphi) = e^{ik\zeta}e^{il\varphi}$. Therefore, the isotropy group of the space $W_k$ is

$$\mathbb{Z}_n(k) = \langle(\zeta, \zeta, -k\zeta)\rangle .$$

# 5 Bifurcation theorem

The orthogonal degree is defined for orthogonal maps that are non-zero on the boundary of some open bounded invariant set. The degree is made of integers, one for each orbit type, and it has all the properties of the usual Brouwer degree. Hence, if one of the integers is non-zero, then the map has a zero corresponding to the orbit type of that integer. In addition, the degree is invariant under orthogonal deformations that are non-zero on the boundary. The degree has other properties such as sum, products and suspensions. For instance, the degree of two pieces of the set is the sum of the degrees. The interested reader may consult [8], [1] and [13] for more details on equivariant degree and degree for gradient maps.

Now, if one has an isolated orbit, then its linearization at one point of the orbit $x_0$ has a block diagonal structure, due to Schur's lemma, where the isotropy subgroup of $x_0$ acts as $\mathbb{Z}_n$ or as $S^1$. Therefore, the orthogonal index of the orbit is given by the signs of the determinants of the submatrices where the action is as $\mathbb{Z}_n$, for $n = 1$ and $n = 2$, and the Morse indices of the submatrices where the action is as $S^1$. In particular, for problems with a parameter, if the orthogonal index changes at some value of the parameter, one will have bifurcation of solutions with the corresponding orbit type. Here, the parameter is the frequency $\nu$.

The fixed point subspace of the isotropy group $\Gamma_{\bar{a}} \times S^1$ corresponds to the block $m_n(0) = B_n$. Since the generator of the kernel is $A_1\bar{a} = T_n(-n^{1/2}e_2)$, then $e_2$ must be in the kernel of $m_n(0)$.

Following [8], one defines $\sigma$ to be the sign of $m_n(0)$ in the orthogonal subspace to $e_2$. Since $B_n = \mu^2 h'(\mu^2) diag(1, 0)$, then

$$\sigma = sgn(e_1^T B_n e_1) = sgn(h'(\mu^2)).$$

We have proved, in [4], that $m_k(0) = B_k$ is invertible except for a point $\mu_k$ for $k = 1, .., n-1$, solution of $\mu^2 h'(\mu^2) = \delta_k$, with

$$\delta_k = (\alpha_k^2 - \gamma_k^2)/(2\alpha_k).$$

**Definition 4** *Following [8], we define*

$$\eta_k(\nu_0) = \sigma\{n_k(\nu_0 - \rho)) - n_k(\nu_0 + \rho)\},$$

*where $n_k(\nu)$ is the Morse index of $m_k(\nu)$.*

This number corresponds to the jump of the orthogonal index at $\nu_0$. Then, from the results of [8], we can state the following theorem for $\mu \neq \mu_1, ..., \mu_{n-1}$.

**Theorem 5** *If $\eta_k(\nu_k)$ is different from zero, then the relative equilibrium has a global bifurcation of periodic solutions from $2\pi/\nu_k$ with isotropy group $\tilde{\mathbb{Z}}_n(k)$.*

The solutions with isotropy group $\tilde{\mathbb{Z}}_n(k)$ must satisfy the symmetries

$$u_{j+1}(t) = e^{\,ij\zeta} u_1(t + jk\zeta).$$

The norms of these oscillators are related by the formula $r_{j+1}(t) = r_1(t + jk\zeta)$, in particular, for $k = n$, all oscillate in an identical fashion. If $k$ divides $n$, then one will have $k$ equal traveling waves, each one formed by $n/k$ oscillators.

By global bifurcation, we mean that the branch goes to infinity in norm or period (for these possibilities we say that the bifurcation is non admissible) or, if not, then the sum of the above jumps, over all the bifurcation points, is zero.

A complete description of these solutions may be found in the paper [5].

# 6 Spectral analysis

The $m_k(\nu)$ have real eigenvalues, so the matrices $m_{n-k}(\nu)$ and $m_k(-\nu)$ have the same spectrum due to the equality $m_{n-k}(\nu) = \bar{m}_k(-\nu)$. As a consequence, the Morse numbers satisfy

$$n_{n-k}(\nu) = n_k(-\nu).$$

For $n = 4$, one has that $\alpha_k = 0$ and the determinant of $m_k(\nu)$ is $-(\gamma_k - \nu)^2$, and there is no jump in the Morse number. Note also that, for $k = n$, one has $\alpha_n = \gamma_n = 0$ and the determinant is $-\nu^2$, that is $m_n(\nu)$ is invertible for $\nu \neq 0$ and there is no bifurcation of periodic solutions with that symmetry.

Since we are going to treat the case $n = 3$ separately, we shall suppose for now that $n \geq 5$. For these values of $n$ one has that $\alpha_k > 0$ and $\delta_k < \alpha_k/2$.

**Proposition 6** *Define $\nu_\pm$ as*

$$\nu_\pm = \gamma_k \pm \sqrt{\alpha_k\left(\alpha_k - 2\mu^2 h'(\mu^2)\right)}.$$

*For $n \geq 5$, the matrix $m_k(\nu)$ changes Morse index only for $\nu_\pm$, whenever $\mu^2 h'(\mu^2) < \alpha_k/2$. In this case*

$$\eta_k(\nu_\pm) = \pm\sigma,$$

*with $\sigma = sgn(h'(\mu^2))$. Furthermore, the values $\nu_\pm$ are positive only in the following cases:*

**(a)** *The value $\nu_+$ is positive for $k \in \{1, ..., n-1\}$ and $\mu^2 h'(\mu^2) < \delta_k$.*

**(b)** *The values $\nu_+$ and $\nu_-$ are positive for $k \in \{1, .., [n/2]\}$ and $\delta_k < \mu^2 h'(\mu^2) < \alpha_k/2$.*

**Proof.** Since the determinant of $m_k$ is

$$d_k(\nu) = \det m_k = -2\alpha_k \mu^2 h'(\mu^2) + \alpha_k^2 - (\gamma_k - \nu)^2,$$

then $d_k(\nu)$ is zero only for $\nu_\pm$, if $\mu^2 h'(\mu^2) < \alpha_k/2$. Furthermore, the trace of $m_k(\nu)$ is

$$T_k = 2\mu^2 h'(|\mu|^2) - 2\alpha_k < -\alpha_k < 0.$$

Since $d_k(\nu)$ is positive in $(\nu_-, \nu_+)$ and negative in the complement, then $n_k(\nu) = 2$ in $(\nu_-, \nu_+)$ and $n_k(\nu) = 1$ in the complement. Thus, $\eta_k(\nu_-) = \sigma(1-2)$ and $\eta_k(\nu_+) = \sigma(2-1)$.

Since $d_k(\nu)$ is a polynomial of degree two, then $\nu_+$ is positive and $\nu_-$ is negative, if $d_k(0)$ is positive. We conclude the result for (a), that is $d_k(0)$ is positive for $\mu^2 h'(\mu^2) < \delta_k$. Furthermore, $d_k(0)$ is negative for $\delta_k < \mu^2 h'(\mu^2)$, thus, $\nu_+$ and $\nu_-$ have the same sign. Since $\nu_\pm = \gamma_k \pm \sqrt{*}$, then $\nu_\pm$ have the sign of $\gamma_k$. We conclude the result for (b), that is $\gamma_k$ is positive for $k \in \{1, .., [n/2]\}$ and that $\gamma_{n-k} = -\gamma_k$. ■

Thus, for the lattice with a general potential for $n \geq 5$ we have:

**Theorem 7** *For each $k \in \{1, ..., n-1\}$ such that $\mu^2 h'(\mu^2) < \delta_k$, the relative equilibrium has a global bifurcation of periodic solutions starting from the period $2\pi/\nu_+$ with symmetries $\tilde{\mathbb{Z}}_n(k)$ . Furthermore, this bifurcation is non-admissible or goes to another equilibrium.*

*For each $k \in \{1, ..., [n/2]\}$ such that $\delta_k < \mu^2 h'(\mu^2) < \alpha_k/2$, the equilibrium has two global bifurcations of periodic solutions starting from the periods $2\pi/\nu_+$ and $2\pi/\nu_-$ with symmetries $\tilde{\mathbb{Z}}_n(k)$.*

**Remark 8** *We have proven that the determinant of the block $m_k(\nu)$ is zero at two values for $k \in \{1, ..., n-1\}$ if $\mu^2 h'(\mu^2) < \alpha_k/2$. Thus, the block $m_k$ corresponds to stable solutions of the linear equation for $\mu^2 h'(\mu^2) < \alpha_k/2$. On the other hand, $\det m_n(\nu)$ has a double zero $\nu = 0$, but the block $m_n$ gives stable solutions due to the spatial symmetries of the problem. Using the fact that $\alpha_k$ are increasing for $k \in \{1, ..., [n/2]\}$, we conclude that the solution $\bar{a}$ is linearly stable for*

$$\mu^2 h'(\mu^2) < \alpha_1/2.$$

The only remaining case is $n = 3$. The proofs are similar to the previous case, by taking the reverse inequalities.

**Proposition 9** *For $n = 3$, the determinant, $\det m_k$, is zero only at $\nu_\pm$ for $\alpha_k/2 < \mu^2 h'(\mu^2)$. In this case one has that $\eta_k(\nu_\pm) = \mp\sigma$ with $\sigma = sgn(h'(\mu^2))$. Furthermore, the values $\nu_\pm$ are positive only in the following cases:*

**(a)** *If $\nu_+$ is positive for $k \in \{1, 2\}$ and $0 < \mu^2 h'(\mu^2)$.*

**(b)** *If $\nu_+$ and $\nu_-$ are positive for $\alpha_1/2 < \mu^2 h'(\mu^2) < 0$.*

**Proof.** Since $\alpha_k < 0$, then $\det m_k$ is zero at $\nu_\pm$ if $\alpha_k/2 < \mu^2 h'(\mu^2)$. In this case the trace of $m_k$ is

$$T_k(\lambda) = 2\mu^2 h'(|\mu|^2) - 2\alpha_k > -\alpha_k > 0.$$

Hence, $n_k(\nu) = 0$ at $(\nu_-, \nu_+)$ and $n_1(\nu) = 1$ on the complement. Thus, $\eta_k(\nu_-) = \sigma(1-0)$ and $\eta_k(\nu_+) = \sigma(0-1)$.

Furthermore, $\nu_+$ is positive when $d_k(0)$ is positive. Since $\alpha_1 = -\gamma_1$, then $\delta_1 = \delta_2 = 0$. We conclude the result for (a) from the fact that $d_k(0)$ is positive for $0 = \delta_k < \mu^2 h'(\mu^2)$ . Also, $d_k(0)$ is negative for $\mu^2 h'(\mu^2) < \delta_k = 0$, then $\nu_+$ and $\nu_-$ have the same sign. Since $\nu_\pm = \gamma_k \pm \sqrt{*}$, then $\nu_\pm$ have the sign of $\gamma_k$. We get the result for (b), from $\gamma_1 = 2\sin^2\zeta > 0$ and that $\gamma_2 = -\gamma_1$. ∎

*Hence, for the lattice and $n = 3$, the relative equilibrium has, for each $k = 1, 2$, if $0 < \mu^2 h'(\mu^2)$, a global bifurcation of periodic solutions starting from the period $2\pi/\nu_+$ with symmetries $\tilde{\mathbb{Z}}_3(k)$.* Furthermore, this bifurcation is inadmissible or goes to another equilibrium.

*For $n = 3$ the relative equilibrium, for $\alpha_1/2 < \mu^2 h'(\mu^2) < 0$, has global bifurcations of periodic solutions starting from the periods $2\pi/\nu_+$ and $2\pi/\nu_-$ with symmetries $\tilde{\mathbb{Z}}_3(1)$.*

For $n = 3$, the relative equilibrium is linearly stable for $\alpha_1/2 < \mu^2 h'(\mu^2)$.

### 6.0.1 Schrödinger potential

For the lattice for the cubic Schrödinger potential, we have $h(\mu) = \mu$ with

$$h'(\mu^2) = 1 \text{ and } \sigma = 1.$$

For $n \geq 5$, one has the following cases:

For $k \in \{1, 2, n-2, n-1\}$ we have $\delta_k \leq 0$, then condition (a) is never satisfied and, for $\mu \in (0, \sqrt{\alpha_k/2})$, condition (b) holds.

For $k \in \{3, ..., n-3\}$ we have $\delta_k > 0$, then, for $\mu \in (0, \sqrt{\delta_k})$, condition (a) is satisfied, while, for $\mu \in (\sqrt{\delta_k}, \sqrt{\alpha_k/2})$, condition (b) holds.

**Theorem 10** *Thus, for the lattice with a Schrödinger potential, for $n \geq 6$, the relative equilibrium has, for each $k \in \{3, ..., n-3\}$ and $\mu \in (0, \sqrt{\delta_k})$, a global bifurcation of periodic solutions, starting from the period $2\pi/\nu_+$ with symmetries $\tilde{\mathbb{Z}}_n(k)$. Furthermore, this bifurcation is inadmissible or goes to another equilibrium.*

*For $n \geq 5$ the relative equilibrium has, for each $k \in \{1, 2\}$ such that $\mu \in (0, \sqrt{\alpha_k/2})$ and, for each $k \in \{3, ..., [n/2]\}$ such that $\mu \in (\sqrt{\delta_k}, \sqrt{\alpha_k/2})$, two global bifurcations of periodic solutions starting from the periods $2\pi/\nu_+$ and $2\pi/\nu_-$ with symmetries $\tilde{\mathbb{Z}}_n(k)$.*

For $n \geq 5$, the relative equilibrium is linearly stable for the amplitudes $\mu \in (0, \sqrt{\alpha_1/2})$.

For $n = 3$ and $k \in \{1, 2\}$, we have that condition (a) is always satisfied. *For $n = 3$, the relative equilibrium has, for each $k \in \{1, 2\}$ such that $\mu \in (0, \infty)$, a global bifurcation of periodic solutions starting from the period $2\pi/\nu_+$ with symmetries $\tilde{\mathbb{Z}}_3(k)$.* Furthermore, for $n = 3$, this relative equilibrium is linearly stable for the amplitudes $\mu \in (0, \infty)$.

#### 6.0.2 Saturable potential

For the lattice with the saturable potential, one has $h(x) = (1 + x)^{-1}$, with

$$h'(\mu^2) = -(1 + \mu^2)^{-2} \text{ and } \sigma = -1.$$

For $n \geq 5$ we have the following cases:

For $k \in \{2, ..., n-2\}$ one has $\delta_k \geq 0$, then, for each $\mu \in (0, \infty)$ one gets $\mu^2 h'(\mu^2) < \delta_k$, that is condition (a).

For $n \in \{16, 17, ...\}$, let $\mu_- \in (0, 1)$ and $\mu_+ \in (1, \infty)$ be the solutions of $\mu^2 h'(\mu^2) = \delta_1$. Then, for $\mu \in (0, \mu_-) \cup (\mu_+, \infty)$, one has $\delta_1 < \mu^2 h'(\mu^2)$, that is condition (b) and, for $\mu \in (\mu_-, \mu_+)$ one has $\mu^2 h'(\mu^2) < \delta_1$, that is condition (a). Furthermore, for $n = \{5, ..., 15\}$ one has the same result as before provided one defines $\mu_-$ and $\mu_+$ to be zero.

**Theorem 11** *Thus, for the lattice with the saturable potential, for $n \geq 5$, the relative equilibrium has, for each $k \in \{2, ..., n-2\}$ and $\mu \in (0, \infty)$ and, for each $k \in \{1, n-1\}$ and $\mu \in (\mu_-, \mu_+)$, a global bifurcation of periodic solutions starting from the period $2\pi/\nu_+$ with symmetries $\tilde{\mathbb{Z}}_n(k)$. This bifurcation is inadmissible or goes to another equilibrium.*

*Moreover, the relative equilibrium has, for each $\mu \in (0, \mu_-) \cup (\mu_+, \infty)$, two global bifurcations of periodic solutions starting from the periods $2\pi/\nu_+$ and $2\pi/\nu_-$ with symmetries $\tilde{\mathbb{Z}}_n(1)$.*

For $n \geq 5$, the relative equilibrium is linearly stable for all amplitudes $\mu \in (0, \infty)$.

For $n = 3$, we have that $\alpha_1/2 = -3/4$, hence, $\alpha_1/2 < -1/4 < \mu^2 h'(\mu^2)$ for all $\mu \in (0, \infty)$. Thus, condition (b) is always satisfied. *For $n = 3$, the relative equilibrium has, for each $\mu \in (0, \infty)$, global bifurcations of periodic solutions starting from the periods $2\pi/\nu_+$ and $2\pi/\nu_-$ with symmetries $\tilde{\mathbb{Z}}_3(1)$.* Furthermore, for $n = 3$, the relative equilibrium is linearly stable for the amplitudes $\mu \in (0, \infty)$.

## References


[1] Z. Balanov, W. Krawcewicz and H. Steinlein, "Applied Equivariant Degree," AIMS Series on Differential Equations & Dynamical Systems, 1. American Institute of Mathematical Sciences (AIMS), 2006.

[2] J. C. Eilbeck and M. Johansson, *The discrete nonlinear Schrödinger equation – 20 years on*, in "Proceedings of the 3rd Conference on Localization

and Energy Transfer in Nonlinear Systems" (editor, Luis Vázquez), 44–67. NJ: World Scientific, Singapore, (2003).

[3] C. García-Azpeitia, "Aplicación del Grado Ortogonal a la Bifurcación en Sistemas Hamiltonianos," UNAM. PhD thesis, 2010.

[4] C. García-Azpeitia and J. Ize, *Global bifurcation of polygonal relative equilibria for masses, vortices and dNLS oscillators*, Journal of Differential Equations, **251** (2011), 3202–3227.

[5] C. García-Azpeitia and J. Ize, *Global bifurcation of planar and spatial periodic solutions from the polygonal relative equilibria for the n-body problem*, Preprint, 2012.

[6] C. García-Azpeitia and J. Ize, *Bifurcation of periodic solutions from a ring configuration in the vortex and filament problems*, Journal of Differential Equations, **252** (2012), 5662–5678.

[7] J. Ize, *Topological bifurcation*, in "Topological Nonlinear Analysis" Progr. Nonlinear Differential Equations Appl., **15**, 341–463. Birkhäuser Boston, (1995).

[8] J. Ize and A. Vignoli, "Equivariant Degree Theory," De Gruyter Series in Nonlinear Analysis and Applications, **8**. Walter de Gruyter, Berlin, 2003.

[9] M. Johansson, *Hamiltonian Hopf bifurcations in the discrete nonlinear Schrödinger trimer: oscillatory instabilities, quasiperiodic solutions and a 'new' type of self-trapping transition*, J. Phys. A: Math. Gen., **37** (2004), 2201–2222.

[10] R. S. MacKay and S. Aubry, *Proof of existence of breathers for time-reversible or hamiltonian networks of weakly coupled oscillators*, Nonlinearity, **7** (1994), 1623–1643.

[11] P. Panayotaros, *Continuation of normal modes in finite NLS lattices*, Phys. Lett. A, **374** (2010), 3912–1919.

[12] C. L. Pando and E. J Doedel, *Bifurcation structures and dominant models near relative equilibria in the one-dimensional discrete nonlinear Schrödinger equation*, Physica D., **238** (2009), 687–698.

[13] S. Rybicki, *Degree for equivariant gradient maps*, Milan J. Math., **73** (2005), 103–144.